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SYMMETRICA, an object oriented computer–algebra system for the symmetric group

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This is a review of an object oriented computer algebra system which is devoted to representation theory, invariant theory and combinatorics of the symmetric group. Moreover, it can be used for classical multivariate polynomials via the different actions of the symmetric group on the algebra of polynomials.

The review contains a brief introduction to the basic methods used. Schubert polynomials are introduced, examples are given, and some applications are described. In particular, they provide a new algorithm for the evaluation of Littlewood–Richardson coefficients via symbolic computations using integer sequences instead of partitions, tableaux or lattice permutations.

1. Introduction

The program system SYMMETRICA is devoted to the linear representation theory, the invariant theory and the combinatorics of finite symmetric groups, as well as symmetrizing operators on the rings of polynomials in several variables. These topics are a vast playground for all kinds of *symbolic calculation* (in the strict sense), since the irreducible representations of the finite symmetric groups $\mathfrak{S}_n$ are parametrized by nonincreasing (or nondecreasing if you prefer the French way of writing partitions) sequences of nonnegative integers, the elements of which sum up to n. The same holds for the conjugacy classes of $\mathfrak{S}_n$, and so such a sequence may stand for an irreducible representation, for a character, for a conjugacy class, or for a Schur polynomial, depending on the problem in question. The symbolic calculations necessary therefore amount to manipulations of partitions.

More generally we now consider arbitrary sequences of natural numbers with finite support (instead of partitions). These sequences index certain polynomials, which recently

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came up (Lascoux and Schützenberger 1982,1985). They are called *Schubert polynomials*, because of their geometric interpretation. Schubert polynomials form a $\mathbb{Z}$-basis of the ring of polynomials

$$\mathbb{Z}[X] := \bigcup_{n>0} \mathbb{Z}[x_1, \ldots, x_n],$$

where the union uses the embedding $\mathbb{Z}[x_1, \ldots, x_n] \hookrightarrow \mathbb{Z}[x_1, \ldots, x_{n+1}]$. Moreover, this basis contains as a subfamily all the Schur polynomials as well as all the monomials with decreasing sequences of exponents (also called the dominant weights).

The combinatorics of Schubert polynomials is very similar to the combinatorics of Schur polynomials. Partitions are replaced by permutations or by the corresponding sequences of nonnegative numbers (Lehmer codes).

The program system SYMMETRICA is based on the use of Schubert polynomials. A by-product is a new algorithm for evaluating the so-called *Littlewood-Richardson coefficients* which play a central rôle in the representation theories of symmetric and of general linear groups. We shall describe this to some detail in the following sections.

The development of SYMMETRICA was started at the Lehrstuhl II für Mathematik University of Bayreuth (the first two of the authors), and at present it is continued in cooperation with the LITP, Université Paris VII (the third author and C. Carré), and the University of Wales at Aberystwyth (T. McDonough and A. O. Morris). Future extensions will incorporate also projective representations and group algebra methods.

2. Schubert polynomials

Consider the polynomial ring $\mathbb{Z}[X] := \mathbb{Z}[x_1, \ldots, x_n]$ and the natural action

$$\mathfrak{S}_n \times \mathbb{Z}[X] \to \mathbb{Z}[X] : (\pi, f) \mapsto f(x_{\pi 1}, \ldots, x_{\pi n}).$$

Hence, for each $f \in \mathbb{Z}[X]$, and every elementary transposition $\sigma_i = (i, i+1)$ in Σ_n, we have a well defined $\sigma_i f \in \mathbb{Z}[X]$, and so we can introduce, for each $i \leq n-1$, the operator ∂_i on $\mathbb{Z}[X]$ by putting

$$\partial_i f := \frac{f - \sigma_i f}{x_i - x_{i+1}}.$$

This operator is called a *divided difference operator*. It was introduced by Newton in interpolation theory. The resulting $\partial_i f$, which is called a *divided difference*, is a polynomial which is symmetric in x_i and x_{i+1} (check this). Moreover, if f is already symmetric in x_i and x_{i+1}, then clearly $\partial_i f = 0$. In the case when f is homogeneous, then $\partial_i f$ is homogeneous, too, it is the zero polynomial, or its degree is 1 less then the degree of f. For example

$$\partial_1 x_1^3 x_2^2 x_3^1 = \frac{x_1^3 x_2^2 x_3^1 - x_1^2 x_2^3 x_3^1}{x_1 - x_2} = x_1^2 x_2^2 x_3.$$

A straightforward check shows that these operators ∂_i satisfy the relations

2.1 $$\partial_i \partial_j = \begin{cases} 0 & \text{if } i = j \\ \partial_j \partial_i & \text{if } |i-j| > 1 \end{cases},$$

and

2.2 $$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}.$$

These relations, which are similar to the Coxeter relations for the generators of the symmetric group (except for $\partial_i\partial_i = 0$) show the link with the symmetric group. They imply the following result:

2.3 Theorem *Lascoux and Schützenberger (1982)*
For any finite sequence $(i_1, \ldots, i_l), i_\nu \in \underline{n-1}$, the following holds:

If $\sigma_{i_1} \ldots \sigma_{i_l} = \sigma_{j_1} \ldots \sigma_{j_l} = \pi^{-1}$ are reduced decompositions, then

$$\partial_{i_1} \ldots \partial_{i_l} = \partial_{j_1} \ldots \partial_{j_l},$$

and hence to each $\pi \in S_n$ there corresponds a unique operator

$$\partial_\pi := \partial_{(i)} := \partial_{i_1} \ldots \partial_{i_l}.$$

If $\sigma_{i_1} \ldots \sigma_{i_l}$ is not *a reduced decomposition, then*

$$\partial_{i_1} \ldots \partial_{i_l} = 0,$$

the zero mapping on $\mathbb{Z}[x_1, \ldots, x_n]$.

□

Now we recall that

$$\omega_n := [n \ldots 1] = (1, n)(2, n-1) \ldots = \omega_n^{-1}$$

is the permutation of maximal length in $\mathfrak{S}_n$: $l(\omega_n) = \binom{n}{2}$. Using this permutation we can associate, according to 2.3, with a permutation $\pi \in \mathfrak{S}_n$ the operator $\partial_{\omega_n \pi}$. We apply this operator to the monomial

$$X^E := X^{E_n} := x_1^{n-1} x_2^{n-2} \ldots x_{n-1}^1,$$

and define in this way the *Schubert polynomial*

$$X_\pi := \partial_{\omega_n \pi} X^E.$$

This construction is compatible with the embedding $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$ as stabilizer of $n+1$. In other words, $X_\mu = X_{\mu'}$, if μ' is obtained from $\mu \in \mathfrak{S}_n$ by adding the fixed point $n+1$. Hence we are really working in $\mathbb{Z}[X]$ instead of a specific $\mathbb{Z}[x_1, \ldots, x_n]$.
For example, if $n := 4$ and $\pi := (243)$, so that $\omega_n \pi = [4132]$, $l(\omega_n \pi) = 4$, and

$$(\omega_n \pi)^{-1} = (34)(12)(23)(34) = \sigma_3 \sigma_1 \sigma_2 \sigma_3$$

is a reduced decomposition. We obtain

$$\begin{aligned} X_{(243)} &= \partial_3 \partial_1 \partial_2 \partial_3 x_1^3 x_2^2 x_3 = \partial_3 \partial_1 \partial_2 \frac{x_1^3 x_2^2 x_3 - x_1^3 x_2^2 x_4}{x_3 - x_4} \\ &= \partial_3 \partial_1 \partial_2 x_1^3 x_2^2 = \partial_3 \partial_1 = \frac{x_1^3 x_2^2 - x_1^3 x_3^2}{x_1 - x_2} \\ &= \partial_3 \partial_1 (x_1^3 x_2 + x_1^3 x_3) = \partial_3 x_1^3 x_2 + x_1^3 x_3 - x_2^3 x_1 - x_2^3 x_3 \\ &= \partial_3 (x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3) = x_1^2 + x_1 x_2 + x_2^2. \end{aligned}$$

Important particular cases are

$$X_{\omega_n} = X^E, \; X_1 = 1.$$

2.4 Lemma *Lascoux and Schützenberger (1982)*
The Schubert polynomials $X_\pi \in \mathbb{Z}[X]$ have the following properties:

The polynomial X_π is homogeneous with nonnegative integral coefficients.
The degree of X_π is equal to the length of π.
In the case when $\pi(i) < \pi(i+1)$, the polynomial X_π is symmetric in x_i and x_{i+1}.
The application of ∂_i has the following effect:

$$\partial_i X_\pi = \begin{cases} X_{\pi\sigma_i} & \text{if } \pi(i) > \pi(i+1) \\ 0 & \text{otherwise.} \end{cases}$$

If $\pi \in \mathfrak{S}_n$ is a permutation π with only one descent, then the Schubert polynomial X_π is a Schur polynomial.

Permutations with only one descent are called *Grassmann-permutations* (Lascoux and Schützenberger, 1985). The corresponding partition which characterizes this Schur polynomial can be obtained by introducing the *Lehmer code* (due in reality to Rothe, 1800!) of a permutation π. This is a sequence of integers $L_1, L_2, \ldots$, where L_i is the number of $j > i$ such that $\pi j > \pi i$. The corresponding mapping L is a bijection between the set of permutations in $\mathfrak{S}_n$ and the integer sequences of the length n, where the i-th entry is not bigger than $n - i$. We can therefore label the Schubert polynomials also by Lehmer codes instead of permutations, writing Y_I instead of $X_{L^{-1}(I)}$. The Grassmann permutations then are exactly the permutations whose Lehmer code I is a increasing sequence of numbers, followed by zeros. If you discard the zeros, you get a *partition*, in French notation. This partition I labels the Schur polynomial, and the number of independents in the alphabet of the Schur polynomial is given by index of the last non-zero entry in I. This index j is the position of the (unique) down in the up-down sequence. So we have the following theorem:

2.5 Theorem *Lascoux and Schützenberger (1985)*
Given a Grassmann permutation π, with its down at position j, then we have

$$X_\pi = S_{L(\pi)}(x_1, \ldots, x_j)$$

where the right side is the notation for a Schur polynomial labelled by the partition $L(\pi)$ in the alphabet $x_1, \ldots, x_j$.

More generally, if j is the index of the first descent of μ, the Schubert polynomial X_μ is symmetric in $x_1, \ldots, x_j$, and the specialization of X_μ, for $x_{j+1} = \ldots = x_n = 0$, is called the *symmetric part* of X_μ.

2.6 Corollary *The Schubert polynomial X_μ is equal to its symmetric part, if and only if μ is Grassmannian, in which case X_μ is a Schur polynomial.*

Let us incidentally remark that the combinatorics of Schur symmetric functions mostly involves manipulations of partitions or of their diagrams (which can be filled and then become Young tableaux). The above theorem shows that *the combinatorics of Schubert polynomials extends the combinatorics of Schur polynomials*, the Lehmer code now replacing partitions. Thus the symbolic calculation with Schubert polynomials amount to the calculation with Lehmer codes.

The curious reader will find now the list of the first 24 Schubert polynomials which correspond to the elements of the symmetric group $\mathfrak{S}_4$:

1234 : [0 , 0 , 0 , 0] : 1
1243 : [0 , 0 , 1 , 0] : $x_3 + x_2 + x_1$
1324 : [0 , 1 , 0 , 0] : $x_2 + x_1$
1342 : [0 , 1 , 1 , 0] : $x_2x_3 + x_1x_3 + x_1x_2$
1423 : [0 , 2 , 0 , 0] : $x_2^2 + x_1x_2 + x_1^2$
1432 : [0 , 2 , 1 , 0] : $x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2$
2134 : [1 , 0 , 0 , 0] : x_1
2143 : [1 , 0 , 1 , 0] : $x_1x_3 + x_1x_2 + x_1^2$
2314 : [1 , 1 , 0 , 0] : x_1x_2
2341 : [1 , 1 , 1 , 0] : $x_1x_2x_3$
2413 : [1 , 2 , 0 , 0] : $x_1x_2^2 + x_1^2x_2$
2431 : [1 , 2 , 1 , 0] : $x_1x_2^2x_3 + x_1^2x_2x_3$
3124 : [2 , 0 , 0 , 0] : x_1^2
3142 : [2 , 0 , 1 , 0] : $x_1^2x_3 + x_1^2x_2$
3214 : [2 , 1 , 0 , 0] : $x_1^2x_2$
3241 : [2 , 1 , 1 , 0] : $x_1^2x_2x_3$
3412 : [2 , 2 , 0 , 0] : $x_1^2x_2^2$
3421 : [2 , 2 , 1 , 0] : $x_1^2x_2^2x_3$
4123 : [3 , 0 , 0 , 0] : x_1^3
4132 : [3 , 0 , 1 , 0] : $x_1^3x_3 + x_1^3x_2$
4213 : [3 , 1 , 0 , 0] : $x_1^3x_2$
4231 : [3 , 1 , 1 , 0] : $x_1^3x_2x_3$
4312 : [3 , 2 , 0 , 0] : $x_1^3x_2^2$
4321 : [3 , 2 , 1 , 0] : $x_1^3x_2^2x_3$

3. Multiplication of Schubert polynomials

For practical purposes an efficient procedure for the multiplication of Schubert polynomials is crucial. Here the following rule, which extends Pieri's formula for Schur polynomials, is very helpful (an elementary proof of it can be found in Kohnert (1987)):

3.1 Monk's Rule *For each X_π and x_k we have*

$$x_kX_\pi = \sum_{\nu=\pi(j,k),j>k,l(\nu)=l(\pi)+1} X_\nu - \sum_{\eta=\pi(j,k),j<k,l(\eta)=l(\pi)+1} X_\eta$$

Using this we can multiply Schubert polynomials, and therefore in particular Schur polynomials. We can also choose k such that only one positive term appears in the product x_kX_π, and so

$$X_\nu = x_kX_\pi + \sum X_\eta.$$

Iterating, we get expansions of the type

$$X_\nu = \sum X_\eta + \sum (x_hx_k\ldots)X_\pi,$$

and we can do this until the summand $\sum X_\eta$ is a sum of Schur polynomial and equal to the symmetric part of X_ν (in which case the second summand is in fact equal to zero). This allows us to compute Littlewood-Richardson coefficents. The corresponding procedure was the starting point for SYMMETRICA , it uses also the following factorization of Schubert polynomials:

3.2 Lemma *Given a permutation $\pi \in S_{n+m}$, which fixes the points $n+1, \ldots, n+m$, and a permutation $\nu \in S_{n+m}$, which leaves each point $1, \ldots, n$ fixed, we have the following factorization property of Schubert polynomials:*

$$X_\pi X_\nu = X_{\pi\nu}$$

Looking at the special case, where the two Schubert polynomials are Schur polynomials, we have solved the problem of multiplying Schur polynomials where the first alphabet is a special subset of the second one. Using now an algorithm, introduced by Lascoux and Schützenberger (1985), which allows us to compute the symmetric part of a Schubert polynomial as a sum of special Schubert polynomials, which are symmetric, and which are Schur polynomials, we can decompose the product of two Schur polynomials in different alphabets into a sum of Schur polynomials. It was shown by Lascoux and Schützenberger (1985), that this decomposition is independent of the size of the alphabet of the two Schur polynomials. This furnishes a decomposition of the product of two Schur functions, giving an algorithm different from the celebrated Littlewood–Richardson Rule (Macdonald, 1979). The algorithm to compute the symmetric part of a Schubert polynomial is part of SYMMETRICA and this routine is used to compute the Littlewood–Richardson coefficients.
More details on Schubert polynomials can also be found in the recently published booklet (Macdonald, 1991)

4. Expansion of multivariate polynomials

The main property of Schubert polynomials is described in

4.1 Theorem *The Schubert polynomials form the $\mathbb{Z}$-basis*

$$\{X_\pi \mid \pi \in \mathfrak{S}_n, \textit{ for some } n \in \mathbb{N}\}$$

of $\mathbb{Z}[X]$, indexed by permutations.

The proof is direct from the following lemma, which shows that, for a suitable order, the matrix of Schubert polynomial in the basis of monomials is a triangular matrix with 1's in the diagonal:

4.2 Lemma *Assume that the monomial $X^D = x_1^{d_1} \cdots x_n^{d_n}$ occurs in $\partial_\pi X^E$ with nonzero coefficient, and suppose that D is the lexicographically smallest sequence of exponents with this property. Then*

The sequence D satisfies the equation

$$D = E - L(\pi) := (n-1-l_1(\pi), \ldots, 1-l_{n-1}(\pi), l_n(\pi)),$$

and
the coefficient of X^D in $\partial_\pi X_\pi$ is equal to 1.

This lemma describes the leading monomial in the Schubert polynomial Y_I, it is the monomial X^I. We can now perform the expansion, which we illustrate by a small example. Consider the polynomial

$$f = x_1x_2^2x_3 + x_1^3x_2.$$

We take the leading term, and we have to compute the corresponding Schubert polynomial. In our example it is (as we see in the list above): $Y_{121} = x_1x_2^2x_3 + x_1^2x_2x_3$. We substract it from f giving a new polynomial with smaller leading term. In our example $-x_1^2x_2x_3 + x^3x_2$. This is a sum of two Schubert polynomials, each of them is a monomial, and hence we have that

$$f = x_1x_2^2x_3 + x_1^3x_2 = Y_{121} - Y_{211} + Y_{31}.$$

This algorithm clearly stops after a finite number of steps.
Hence we can do *symbolic calculation* with Schubert polynomials by using an indexing with vectors in $\mathbb{N}^\infty$, having only a finite number of nonzero components.

5. Further Parts

Further parts of SYMMETRICA are routines for character evaluations and character decompositions. The character tables of symmetric groups can be evaluated as well as single values of characters. For these purposes the recursion procedure called *Murnaghan–Nakayama Rule* is implemented, but also the *method of characteristics* which uses a generating function for the values of an irreducible character. This method turns out to be much faster for higher degrees of the symmetric groups. The decomposition of inner products of characters uses the direct method and the corresponding character table. The decomposition of outer products uses the Littlewood–Richardson coefficients and their evaluation via the multiplication of Schubert polynomials as it is described above (and not the classical version via skew tableaux, where a check is necessary, if the skew tableau is a "lattice permutation", or not).
There is also a procedure that gives the character tables of wreath products of symmetric groups, using the so-called *method of characteristics* (Specht, 1932). The character tables together with further matrices which are also provided allow to switch between different bases of symmetric functions (Macdonald, 1979). Another routine allows to calculate generalized matrix functions.
Additional procedures yield reduced decompositions of permutations, via an application of the *Lehmer code.*
A further tool is a very fast and far reaching procedure for the evaluation of plethysm of identity representations (Carre, 1990).
The approach to the modular irreducible matrix representations (Golembiowski, 1987) uses the *standard bideterminants* according to (Clausen, 1980). It covers also the ordinary case but there is also a special program contained which gives the integral form of the ordinary irreducible matrix representations only, namely *Young's Natural Form,* and which therefore is much faster. There are in fact two slightly different integral forms of the ordinary irreducible representations available: the version described in James and Kerber (1981), as well as the version described in Boerner (1955).
Emphasize is laid also on the method of *symmetry adapted bases,* which is the main tool for applications to sciences. There is a procedure using symmetry adapted bases in order to evaluate the ordinary irreducible polynomial representations of the general

linear group. And there is also a routine that allows to transform an operator with a given symmetry as soon as the ordinary irreducible representations of the symmetry group in question are at hand.
Also the first steps towards calculations in the group algebra of the symmetric group are made, minimal idempotents can be evaluated etc.
Due to a program system which was contributed by T. McDonough cyclotomic arithmetic can be done, square radical extension of the rational field are at hand. Therefore also the character tables of alternating groups are available.

6. Future developments

In the near future the matrix representations of alternating groups will be incorporated, as well as parts of the theory of *projective representations* of symmetric groups (contributed mainly by T. McDonough and A. O. Morris, Aberystwyth). Moreover, we shall extend the procedures that allow to deal with the group algebra (in cooperation with W. Müller, Bayreuth, and G. M. and G. E. Murphy, London).
The ring of polynomials, modulo the ideal generated by symmetric polynomials without constant term is isomorphic to the regular representation of the symmetric group. Some ideals in this ring have a geometric interpretation in cohomology theory of flag manifolds. Schubert polynomials and divided differences allow to test whether an element belong to one of these ideals or not (cf. Akiylidiz, Lascoux and Pragacz, 1989), instead of using such tools as Gröbner bases.
Interpolation in several variables involve summations generalizing Weyl character formula, Schubert polynomials in two sets of variables provide such summations, and for example, are the coefficients of divided differences in the Newton interpolation formula in several variables. Therefore, SYMMETRICA will include in the near future several functions needed in geometry (Chern classes), in interpolation theory (Newton interpolation in several variables ...), in classical algebra (straightening formula ...) and in algebraic combinatorics (cycle indicator polynomials and generating functions for actions of finite groups on finite sets, see Kerber (1991)).

7. How to get SYMMETRICA

The program is free for non commercial purposes, if you are interested, please send a 3.5″ HD-diskette to the first author, and we shall put on it the C-source code of programs together with a Latex-file of a "User's Guide". We just require that you send back the relevant procedures that you want us to add to the system, and that you tell us about your applications of SYMMETRICA .
The program system can also be obtained through *anonymous ftp,* from

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